Noise-induced sensitivity to the initial conditions in stochastic dynamical systems

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It is shown that the occurrence of a positive time-averaged Lyapunov exponent in a nonlinear system subject to noise, be it additive or multiplicative, does not necessarily imply deterministic chaos.

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In Ref. [1], it is pointed out that the initial separation of trajectories in dynamical systems linearized around a stable fixed point and subjected to certain types of additive noise is described by a power law rather than the exponential separation observed in deterministic chaos. In this sense, the outcome in such a noisy system is more predictable than that of a deterministic chaotic system. It is also pointed out that a system with multiplicative noise can be characterized by a positive Lyapunov exponent even when the underlying deterministic dynamics does not exhibit chaos. In this Brief Report we want to point out that such a situation can also arise in a system with purely additive noise, provided that the full nonlinearity of the dynamics is taken into account. The observation of a positive time-averaged Lyapunov exponent in a nonlinear system subject to noise, be it additive or multiplicative, does therefore not imply deterministic chaos.

To illustrate this point we consider the following model:

$$\frac{\partial x}{\partial t}=-U'(x)+\xi \ , \tag{1}$$

where ξ is a Gaussian white noise of intensity D, and $U(x)$ is a potential function. The invariant density $P^{\rm st}(x)$ for the stochastic process x that obeys the stochastic differential Eq. (1) is given by (assuming ergodicity)

$$P^{\rm st}(x)\sim\exp\left[-\frac{U(x)}{D}\right] . \tag{2}$$

Consider now the separation $\Delta x(\tau)=x_2-x_1$ of two trajectories starting close to the initial point x (but subject to two independent realizations of the Gaussian white-noise process ξ). For small times τ, $\Delta x(\tau)$ corresponds to an Ornstein-Uhlenbeck process obeying the following stochastic differential equation:

$$\frac{\partial \Delta x}{\partial t}=-U''(x)\Delta x+\sqrt{2}\xi \ . \tag{3}$$

The quantity of interest is the evolution of the absolute value of the separation $|\Delta x(\tau)|$, starting from the initial separation $|\Delta x(0)|=\epsilon$. From Eq. (3), we get the following result:

$$\langle|\Delta x(\tau)|\rangle=\sqrt{8D\tau}\left[\frac{1}{\sqrt{\pi}}\exp(-\rho^2)+\rho\,{\rm erf}(\rho)\right] , \tag{4}$$

with

$$\rho=\frac{\epsilon\exp[-U''(x)\tau]}{\sqrt{8D\tau}} . \tag{4a}$$

The error function of x is represented by erf(x). The result Eq. (4) only holds for times τ and initial separations ϵ small enough such that the linearized stochastic dynamics given by Eq. (3) is valid. Restricting ourselves to this situation, we note that two different regimes can be distinguished in the behavior of $|\Delta x(\tau)|$ depending on the value of ρ. For $\rho\ll 1$ the separation of trajectories is dominated by the noise and is equal to that of two independent Brownian trajectories. One thus recovers the power law mentioned in [1],

$$\langle|\Delta x(\tau)|\rangle=\sqrt{8D\tau/\pi} . \tag{5}$$

This result corresponds to a subexponential separation of the trajectories, and the corresponding value of the Lyapunov exponent is equal to zero. Note that the condition $\rho\ll 1$ is compatible with the usual definition of a Lyapunov coefficient in which the limit $\epsilon\to 0$ has to be taken prior to the limit $\tau\to 0$. It is however conceivable that one does not possess the required accuracy on x to reach this regime, and the consideration of the other regime becomes relevant. In this case, $\rho\gg 1$, the drift term dominates in Eq. (3), and Eq. (4) predicts that trajectories converge or diverge exponentially, depending on the value of the local curvature of the potential,

$$\langle|\Delta x(\tau)|\rangle=\epsilon\exp[-U''(x)\tau] . \tag{6}$$

The local Lyapunov exponent, characterizing this exponential separation, is given by $\lambda(x)=-U''(x)$. Assuming ergodicity, we conclude that the time-averaged Lyapunov exponent reads

$$\lambda=\int dx\,\lambda(x)P^{\rm st}(x)=-\int dx\,U''(x)P^{\rm st}(x) , \tag{7}$$

where the integrals run over the support of the steady-state probability $P^{\rm st}(x)$ (which is typically from $-\infty$ to $+\infty$). Regions of the potential with a negative curvature

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give a positive contribution to the overall value of the Lyapunov exponent λ. It is quite possible that such regions dominate even when the potential $U(x)$ has a unique globally stable minimum.

A simple example of a potential that illustrates the above possibilities is schematically represented in Fig. 1. An analytical study of this model, taking into account the effect of all the small parameters involved, would clearly be desirable, but it is very involved. We rather give a qualitative discussion of how, in this example, the value and sign of the Lyapunov exponent will change with the intensity of the noise. We assume that we are using ϵ and τ values such that the linearized equation (3) is valid and that $\rho \gg 1$. These conditions can always be fulfilled by choosing ϵ and τ sufficiently small and the ratio ϵ/τ large enough. For a small value of the noise intensity D, the probability density $P^{st}(x)$ will be concentrated in a small region around $x=0$ where $U''(x)>0$. We will thus observe an exponential convergence of the trajectories with a negative value of the Lyapunov exponent λ; cf. Eq. (7). As the noise intensity D increases, while the value of ρ is still much larger than 1, the region further away from $x=0$, where $U''(x)<0$, will get more weight and eventually dominate in the integral given in Eq. (7). A positive value of the Lyapunov coefficient λ results. For even larger values of D, the value of ρ will further decrease and we will leave the region where an exponential separation between trajectories is observed. Eventually for $\rho \ll 1$ we will observe the subexponential separation, cf. Eq. (5), with the corresponding value of the Lyapunov exponent equal to zero.

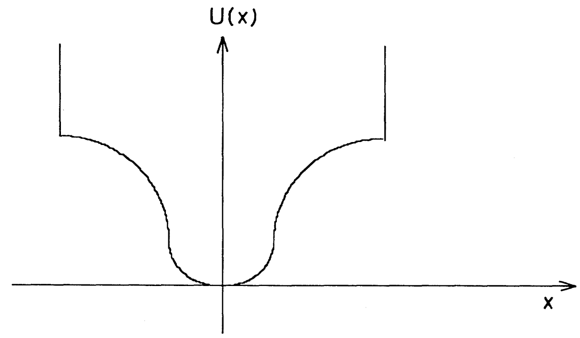

FIG. 1. Schematic representation of a potential $U(x)$ for which the Lyapunov exponent λ defined in Eq. (7) will change with the amplitude of the noise.

The above example illustrates that the value or even the sign of the effective Lyapunov exponent λ, obtained in nonlinear systems with noise, may not reflect the properties of the underlying deterministic attractor.

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[1] K. Seki, V. Balakrishnan, and G. Nicolis, Phys. Rev. E **47**, 155 (1993).